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SIMPLE WAVES ON A SHEAR GAS FLOW IN A CHANNEL OF CONSTANT CROSS SECTION

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The plane-parallel unsteady-state shear gas flow in a narrow channel of constant cross section is considered. The existence theorem of solutions in the form of simple waves of a set of equations of motion is proved for a class of isentropic flows with a monotone velocity profile over the channel depth. The exact solution described by incomplete beta-functions is found for a polytropic equation of state in a class of isentropic flows.

Introduction. An approximate integrodifferential system of equations of nonstationary flows of an inviscid heat-nonconducting gas in an elongated channel of variable cross section was derived by Teshukov [1]. Based on the generalized determination of the characteristics and the notion of hyperbolicity for systems with operator functionals [2, 3], Teshukov [1] obtained hyperbolicity conditions for a system of equations of motion and constructed a class of stationary solutions which describe inhomogeneous transonic flows in a channel of variable cross section.

Exact solutions of a system of equations of long waves propagating in a layer of incompressible vortex fluid were found by Freeman [4] and Blythe et al. [5]. Teshukov [6] and the author [7] showed the existence of simple waves and analyzed their common properties which correspond to particular values of the characteristic spectrum in a free-boundary fluid layer.

1. Formulation of the Problem. We consider the initial boundary-value problem

$$\begin{gathered} u_T + uu_X + vu_Y + \rho^{-1}p_X = 0, \quad \rho^{-1}p_Y = 0, \quad \rho_T + u\rho_X + v\rho_Y + \rho(u_X + v_Y) = 0, \\ s_T + us_X + vs_Y = 0, \quad u(X,0,Y) = u_0(X,Y), \quad v(X,0,Y) = v_0(X,Y), \\ s(X,0,Y) = s_0(X,Y), \quad \rho(X,0,Y) = \rho_0(X,Y), \quad \rho = \rho(p,s), \quad 0 \leqslant Y \leqslant A_0(X), \\ X \in R, \quad T \in R^+, \quad v(X,T,0) = 0, \quad v(X,T,A_0(X)) = u(X,T,A_0(X))A_0'(X), \end{gathered} \tag{1.1}$$

which describes the plane-parallel vortex gas flow in a channel $0 \leqslant Y \leqslant A_0(X)$ in a long-wave approximation. Here u and v are the velocity-vector components, p is the pressure, ρ is the density, and s is the entropy. Hereafter, the specific volume ρ^{-1} is denoted by τ and it is assumed that the equation of state of the gas $p(\tau, s)$ satisfies the conditions

$$p_\tau < 0, \qquad p_{\tau\tau} > 0, \qquad p_s > 0, \qquad \rho \to 0 \quad \text{as} \quad p \to 0. \tag{1.2}$$

The long-wave approximation arises if one takes into account that $H_0 \ll L_0$, where H_0 and L_0 are the characteristic depth and length of the channel. It follows from the second equation of (1.1) that the pressure does not depend on the vertical coordinate Y: $p = p(X,T)$. This means that the pressure equalizes instantly across the channel in the long-wave approximation.

We introduce the mixed Eulerian–Lagrangian independent variables x, t, and λ [$X = x$, $T = t$, $Y = \Phi(x,t,\lambda)$, $0 \leqslant \lambda \leqslant 1$, $\Phi(x,t,0) = 0$, and $\Phi(x,t,1) = A_0(x)$] [1] and the new desired function

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$H(x,t,\lambda) = \rho(x,t,\lambda)\Phi_\lambda(x,t,\lambda)$. According to [1], problem (1.1) is reduced to the Cauchy problem for the functions $u(x,t,\lambda)$, $H(x,t,\lambda)$, and $s(x,t,\lambda)$:

$$u_t + uu_x + (\rho\sigma)^{-1}\int_0^1 (H_x\rho^{-1} - H\rho_s s_x\rho^{-2})\,d\lambda = A_0'(x)(\sigma\rho)^{-1}, \quad H_t + (uH)_x = 0, \quad s_t + us_x = 0,$$

$$u(x,0,\lambda) = u_0(x,\lambda), \qquad H(x,0,\lambda) = H_0(x,\lambda), \qquad s(x,0,\lambda) = s_0(x,\lambda), \tag{1.3}$$

$$\rho = \rho(p(x,t), s(x,t,\lambda)), \qquad 0 \leqslant \lambda \leqslant 1, \quad -\infty \leqslant x \leqslant +\infty, \quad t > 0,$$

where $\sigma = \int_0^1 H\rho^{-2}c^{-2}\,d\lambda$ and c^2 is the squared sound velocity ($c^2 = p'_\rho$). Here the nonlocal dependence of p on s and H in (1.3) is set by the equation

$$\int_0^1 H(x,t,\lambda)(\rho(p(x,t), s(x,t,\lambda)))^{-1}\,d\lambda = A_0(x). \tag{1.4}$$

In accordance with the known solution (1.3), the replacement function $\Phi(x,t,\lambda)$ and the vertical component of the velocity vector v are determined by the relations

$$\Phi(x,t,\lambda) = \int_0^\lambda H(x,t,\nu)(\rho(p(x,t), s(x,t,\nu)))^{-1}\,d\nu, \qquad v = \Phi_t + u\Phi_x.$$

According to [1], system (1.3) has the characteristics $dx/dt = k_i(x,t)$, which correspond to the discrete spectrum, and the characteristics $dx/dt = u(x,t,\lambda)$ ($\lambda = \text{const}$) of the continuous spectrum of the characteristic velocities. The characteristic roots of the discrete spectrum are determined by the equation

$$\sigma = \int_0^1 H\rho^{-2}(u - k_i)^{-2}\,d\lambda, \tag{1.5}$$

which has only two real roots k_1 and k_2 outside the range of variation of the function $u(x,t,\lambda)$ such that $k_1 < \min\limits_\lambda u(x,t,\lambda)$ and $k_2 > \max\limits_\lambda u(x,t,\lambda)$ for any values of the variables x and t.

We consider a channel of constant cross section $A_0(x) = A_0 = \text{const}$. We note that system (1.3) admits exact solutions of the form

$$p = p_0 = \text{const}, \qquad u = u(\lambda), \qquad H = H(\lambda), \qquad s = s(\lambda); \tag{1.6}$$

$$u = u(\eta(x,t),\lambda), \qquad H = H(\eta(x,t),\lambda), \qquad s = s(\eta(x,t),\lambda), \tag{1.7}$$

where $\eta(x,t)$ is a certain function of the variables x and t. In initial variables, solution (1.6) describes a steady-state shear flow $u = u(Y)$, $v = 0$, $p = p_0 = \text{const}$, and $s = s(Y)$, and (1.7) gives the solution $u = u(\eta(X,T),Y)$, $v = \eta_X v(\eta(X,T),Y)$, and $s = s(\eta(X,T),Y)$, which we call a *simple* wave. In what follows, we use the function of pressure distribution $\eta(x,t)$ as the simple-wave parameter $p(x,t)$.

Here we consider simple waves which satisfy the following condition for all λ:

$$k = -p_t/p_x \neq u(x,t,\lambda). \tag{1.8}$$

By virtue of (1.8), the simple waves are described by the equations

$$u_p = -(\rho(p, s(p,\lambda)))^{-1}(u-k)^{-1}, \qquad H_p = H(\rho(p, s(p,\lambda)))^{-1}(u-k)^{-2}, \quad s_p = 0. \tag{1.9}$$

Having divided the second equation of (1.9) by ρ and integrated it over λ from 0 to 1, according to (1.4) we obtain that k should satisfy the characteristic equation (1.5). For definiteness, we analyze simple waves which correspond to the root k_2 [$k_2 > \max\limits_\lambda u(x,t,\lambda)$] of Eq. (1.5) (the case $k = k_1$ is considered similarly). It is convenient to derive a differential equation for k by differentiating (1.5):

$$k'(p) = -\Big(\int_0^1 H\tau_{pp}\,d\lambda + 3\int_0^1 H\tau(u-k)^{-2}(\tau_p + \tau^2(u-k)^{-2})\,d\lambda\Big)\Big/2\int_0^1 H\tau^2(u-k)^{-3}\,d\lambda. \tag{1.10}$$

For system (1.9), (1.10), it is natural to pose the Cauchy problem with data for $p = p_0$ (p_0 = const):

$$u(p_0,\lambda) = u_0(\lambda), \quad H(p_0,\lambda) = H_0(\lambda), \quad s(p_0,\lambda) = s_0(\lambda), \quad k(p_0) = k_0. \tag{1.11}$$

Here k_0 is the large root of Eq. (1.5) [$k_0 > \max u_0(\lambda)$] for $u = u_0(\lambda)$, $H = H_0(\lambda)$, and $s = s_0(\lambda)$.

According to solution (1.9), (1.10), the pressure p is found by integration of (1.8). It follows from (1.8) that the pressure p is constant along the characteristics $dx/dt = k_2$ of system (1.3). Thus, the range of definition of a simple wave is covered with the one-parameter family of planes p = const, and problem (1.9)–(1.11) is the problem of contiguity of a simple wave-type continuous solution to the specified shear flow with respect to a certain characteristic which corresponds to $p = p_0$ (p_0 is the constant pressure in the shear flow).

2. Existence of Simple Waves. Proving the existence of simple waves, we rely on the existence theorem of a solution of the Cauchy problem for a nonlinear equation in the Banach space B:

$$\frac{dx}{dt} = f(x,t), \qquad x(t_0) = x_0. \tag{2.1}$$

Here $f(x,t)$ is the function of real argument t and variable $x \in B$ which takes on the values in B.

Let the function $f(x,t)$ be continuous with respect to t and satisfy the conditions $\|f(x,t)\| \leqslant M_1$ and $\|f(x_1,t) - f(x_2,t)\| \leqslant M_2\|x_1 - x_2\|$ for $t \in [a,b]$ and $\|x - x_0\| \leqslant \theta$. According to [8], there is $\delta_1 > 0$ such that the Cauchy problem (2.1) has a unique solution $x = \varphi(t)$ which is left in the sphere $\|\varphi(t) - x_0\| \leqslant \theta$ on the interval $\delta_1 = \min(\theta M_1^{-1}, M_2^{-1})$.

To use this result, we consider the Banach space B of the vector functions $\mathbf{U} = (u,H,s,k)$ of real argument $\lambda \in [0,1]$

$$B = \{(u,H,s,k)/u \in C^1[0,1],\ H \in C[0,1],\ s \in C^1[0,1],\ k \in R\}$$

with the norm $\|\mathbf{U}\| = \max|u_\lambda| + \max|u| + \max|s_\lambda| + \max|s| + \max|H| + |k|$, where $C^1[0,1]$ is the set of continuously differentiable functions on the segment $[0,1]$, $C[0,1]$ is the set of continuous functions, and R is the numerical straight line.

Let $\mathbf{U}_0 = (u_0, H_0, s_0, k_0) \in B$. Since u_0 and H_0 are continuous in the closed gap $[0,1]$ and $u_0 - k_0 < 0$ and $H_0 > 0$, there is a constant $\theta > 0$ such that $|u_0 - k_0| \geqslant \min|u_0(\lambda) - k_0| > \theta$ and $\min H_0 > \theta$. We consider a sphere $\|\mathbf{U} - \mathbf{U}_0\| < \theta/2$ in the space B. For $\mathbf{U}$ from the sphere, the following inequalities are fulfilled:

$$|u-k| \geqslant |u_0 - k_0| - \|\mathbf{U} - \mathbf{U}_0\| \geqslant \theta/2, \quad \min|H| \geqslant \min|H_0| - \|\mathbf{U} - \mathbf{U}_0\| \geqslant \theta/2. \tag{2.2}$$

By virtue of the continuity of the operator $f(\mathbf{U},p)$, there are constant $M_1(\theta,\mathbf{U}_0)$ and $M_2(\theta,\mathbf{U}_0)$ in the region (2.2) for which the inequalities

$$\|f(\mathbf{U},p)\| \leqslant M_1, \qquad \|f(\mathbf{U}_1,p) - f(\mathbf{U}_2,p)\| \leqslant M_2\|\mathbf{U}_2 - \mathbf{U}_1\| \tag{2.3}$$

hold true. Using the above result, we establish the fact of the existence and uniqueness of the solution of problem (1.9)–(1.11) on the interval $[p_0 - \delta_1, p_0 + \delta_1]$ in B.

For isentropic flows, for $s_0(\lambda) = s_0$ = const we prove the existence theorem of a solution of problem (1.9)–(1.11) as a whole with the use of the amplitude of the simple wave p. We consider the monotone velocity profile $u_{0\lambda} \geqslant 0$. It follows from (1.9) that $(u_\lambda H^{-1})_p = 0$. Then the relation

$$u_\lambda H^{-1} = u_{0\lambda} H_0^{-1} \tag{2.4}$$

is the integral of system (1.9), (1.10). By virtue of (2.4), we have $u_\lambda > 0$ in the range of definition of a simple wave. In addition, the characteristic equation (1.5) is the integral of Eqs. (1.9) and (1.10) by construction. The use of the integrals of system (1.9), (1.10) allows one to estimate the solution *a priori*.

Lemma. *Let u, H, and k be the solution of problem (1.9)–(1.11) and the inequality $\omega = u_{0\lambda}H_0^{-1} \leqslant \omega_2 < \infty$, where $\omega_2 = \max \omega(\lambda)$, be satisfied. Then the estimates*

$$\frac{c^2}{c+\omega_2\rho A_0} \leqslant |u-k| \leqslant A_0\omega_2\rho + c, \tag{2.5}$$

where $c = \sqrt{p'_\rho}$ is the velocity of sound, are valid.

Proof. We introduce $u_2 = \max\limits_\lambda u(p(x,t),\lambda)$ and $u_1 = \min\limits_\lambda u(p(x,t),\lambda)$. It follows from the characteristic equation (1.5) that

$$A_0\rho^{-1}c^{-2} = \sigma = \int_0^1 H\rho^{-2}(u-k)^{-2}\,d\lambda \leqslant A_0\rho^{-1}(u_2-k)^{-2},$$

$$A_0\rho^{-1}c^{-2} = \sigma = \int_0^1 H\rho^{-2}(u-k)^{-2}\,d\lambda \geqslant A_0\rho^{-1}(u_1-k)^{-2}.$$

Whence

$$|u_2-k| \leqslant c, \qquad |u_1-k| \geqslant c. \tag{2.6}$$

From (1.5), we obtain

$$A_0\rho^{-1}c^{-2} \geqslant \rho^{-2}\omega_2^{-1}\int_0^1 u_\lambda(u-k)^{-2}\,d\lambda = \rho^{-2}\omega_2^{-1}(-(u_2-k)^{-1}+(u_1-k)^{-1}). \tag{2.7}$$

By virtue of (2.6), the estimate from below follows from (2.7): $|u_2-k| \geqslant c^2(c+A_0\omega_2\rho)^{-1}$. From (1.4), we obtain

$$A_0 = \rho^{-1}\int_0^1 u_\lambda^{-1}Hu_\lambda\,d\lambda \geqslant (\rho\omega_2)^{-1}(u_2-u_1). \tag{2.8}$$

Since $|u_1-k| \leqslant u_2-u_1+|u_2-k|$, the upper estimate follows from (2.6) and (2.8): $|u_1-k| \leqslant A_0\omega_2\rho + c$. The inequalities (2.5) follow from the inequalities $|u_2-k| \leqslant |u-k| \leqslant |u_1-k|$. The lemma is proved.

Theorem. Let u and H satisfy the conditions of the lemma. Then the solution of problem (1.9)–(1.11) exists on any finite interval $p \in [\delta, L]$, where $\delta > 0$ and $L < \infty$ and belongs to the space B.

Proof. For $\mathbf{U} = (u,H,k) \in B$ and $p \in [\delta, L]$, by virtue of (2.5) the inequalities

$$c^2(\delta)(c(L)+\omega_2\rho(L)A_0)^{-1} \leqslant |u-k| \leqslant A_0\omega_2\rho(L)+c(L) \tag{2.9}$$

hold true. Differentiating the first equation of (1.9) with respect to λ and integrating over p, we obtain

$$u_\lambda(p,\lambda) = u_{0\lambda}\exp\Big(\int_{p_0}^p \rho^{-1}(u-k)^{-2}\,dp\Big). \tag{2.10}$$

The integration of the second equation in (1.9) yields

$$H(p,\lambda) = H_0(\lambda)\exp\Big(\int_{p_0}^p \rho^{-1}(u-k)^{-2}\,dp\Big). \tag{2.11}$$

According to (2.9)–(2.11), conditions (2.2) are satisfied with the same constants M_1 and M_2, which depend only on δ, L, and $\|\mathbf{U}_0\|$. Therefore, after the solution is constructed on the interval $[p_0-\delta_1, p_0+\delta_1]$, it can be extended uniquely over the entire interval $[\delta, L]$. The theorem is proved.

The construction of the simple wave is completed by solving the equation

$$p_t + k(p)p_x = 0, \qquad p(x,0) = p_m(x). \tag{2.12}$$

According to the known facts of the theory of quasilinear equations [9], the properties of solutions (2.12) depend on whether the derivative $k'(p)$ is of fixed sign or not. It follows from Eq. (2.12) that

$$p_x = \frac{p_{mx}(x)}{1 + tk'(p_m)p_{mx}} \tag{2.13}$$

along the characteristics $dx/dt = k(p)$. It follows from (2.13) that if $k'(p_m)p_{mx} > 0$, the derivative p_x remains bounded ($|p_x| \leqslant 1$) and the solution of Eq. (2.12) exists for any $t > 0$. If $k'(p_m)p_{mx} < 0$ for certain x, the solution of (2.12) exists only for finite $t > 0$.

We examine the sign of the function $k'(p)$. We note that the numerator of the right-hand side of Eq. (1.10) coincides with the form of the second derivative $K''(p)$ of the functions $K(p)$, which serves to determine the pressure p in constructing steady-state solutions in [1]. Here the function f from [1] corresponds to the functions H. The condition $K'(p_c) = 0$ from [1] is equivalent to the characteristic equation (1.5), and the condition $K''(p_c) \geqslant 0$ to the condition $k'(p) \geqslant 0$. As is shown in [1], if the equation of state has the form

$$\tau = B(s)\varphi(p), \qquad \varphi' < 0, \qquad \varphi'' > 0, \tag{2.14}$$

we have $k'(p) \geqslant 0$. Generally, the condition $k'(p) > 0$ is not a consequence of conditions (1.2); however, if one requires that the function of the equation of state $\tau = \tau(p, s)$ satisfy the additional condition

$$4\tau\tau_{pp} - 3\tau_p^2 > 0, \tag{2.15}$$

we have $k'(p) > 0$ for any p.

We call the simple wave a *compression* (*rarefaction*) wave if the inequality

$$p_t + u(x, t, \lambda)p_x = (u(x, t, \lambda) - k)p_x > 0 \quad (< 0)$$

is satisfied for any λ ($0 \leqslant \lambda \leqslant 1$).

According to the equations of gas dynamics, we call the simple wave a *centered* wave if all characteristics $dx/dt = k(p)$ converge at one point.

It follows from (2.13) that the gradient catastrophe will occur in a simple compression wave for $k'(p) > 0$, and simple rarefaction waves can exist for any $t > 0$. If $k'(p) < 0$, simple compression waves can exist for any $t > 0$, and rarefaction waves only for finite $t > 0$. Centered at the initial moment, simple waves are rarefaction waves for $k'(p) > 0$ and compression waves for $k'(p) < 0$.

The equation of state of a gas with constant entropy is a particular case of (2.14). Hence, $k'(p)$ are always greater than zero for isentropic flows, and simple waves defined for all $t > 0$ are the rarefaction waves and the simple compression waves collapse. Centered at $t = 0$, simple waves for $t > 0$ are the rarefaction waves.

We mention an analogy. The behavior of a simple wave-type solution of the equations of one-dimensional gas dynamics is determined by the sign of the quantity $g_{\tau\tau}$, where $p = g(\tau,\ s)$ is the equation of state of a gas. In studying the simple waves of system (1.3), this role is played by the sign of the derivative $k'(p)$. Therefore, one can regard condition (2.15), which guarantees that $k'(p) > 0$, as an analog of the condition of convexity of the equation of state: $g_{\tau\tau} > 0$.

Thus, a simple isentropic rarefaction wave exists for any $t > 0$. Generally, simple rarefaction and compression waves can exist, depending on the initial equation of state. The behavior of the solution is determined by the sign of the derivative $k'(p)$, and the properties of simple waves, when $k'(p)$ changes sign, are similar to the properties of simple waves in a gas with anomalous thermodynamic properties.

3. Exact Solutions. We consider the problem of construction of a simple wave (1.9), (1.10) for a medium with a polytropic equation of state $\rho = B_0 p^\beta$, where $0 < \beta < 1$, and constant entropy $B_0 = \text{const}$. We denote $\alpha = \beta/(1-\beta)$ and introduce the new variable $\xi = p^{1-\beta}/(1-\beta)$. Equations (1.9) and relation (1.4) can be rewritten in the form

$$u_\xi = -(B_0(u-k))^{-1}, \qquad H_\xi = HB_0^{-1}(u-k)^{-2}, \qquad \int_0^1 H\, d\lambda = B_0 A_0 (1-\beta)^\alpha \xi^\alpha. \tag{3.1}$$

System (3.1) admits a one-parameter group of extensions $\xi \to l\xi$, $\lambda \to \lambda$, $u \to \sqrt{l}u$, $k \to \sqrt{l}k$, and $H \to l^{\alpha}H$, where l is a parameter of the group. We construct a solution of system (3.1) which is invariant relative to this group. We note also that the ratio

$$y = \int_0^{\lambda} H\,d\lambda \Big/ \int_0^{1} H\,d\lambda = \int_0^{\lambda} H\,d\lambda \Big/ B_0 A_0 (1-\beta)^{\alpha} \xi^{\alpha} = \frac{Y}{A_0}$$

is the invariant of the group. According to the known algorithm of searching for invariant solutions, we assume that

$$u = \sqrt{\xi}U(y), \qquad k = \sqrt{\xi}A \quad (A = \text{const}). \tag{3.2}$$

Substituting (3.2) into the first equation of (3.1), we obtain an equation to define the function $U(y)$:

$$U/2 + (\Phi - \alpha y)U' + (B_0(U-A))^{-1} = 0, \tag{3.3}$$

where

$$\Phi(y) = \int_0^{y} B_0^{-1}(U-A)^{-2}\,dy. \tag{3.4}$$

By virtue of (1.5) and (3.4), the functions $\Phi(y)$ satisfy the boundary conditions $\Phi(0) = 0$ and $\Phi(1) = \alpha$. Excluding $U(y)$ from (3.3) and using (3.4), we obtain the following equation for $\Phi(y)$:

$$-(\Phi - \alpha y)\Phi'' + 2\Phi'^2 + A\sqrt{B_0}\Phi'^{3/2} + \Phi' = 0.$$

We introduce the function $\psi = \Phi - \alpha y$. Then $\psi(y)$ should satisfy the equation

$$-\psi\psi'' + 2(\psi' + \alpha)^2 + A\sqrt{B_0}(\psi' + \alpha)^{3/2} + \psi' + \alpha = 0 \tag{3.5}$$

and the boundary conditions $\psi(0) = \psi(1) = 0$. After the replacements $\psi' = L(\psi)$ and $N^2 = L + \alpha$, Eq. (3.5) is integrated similarly [4]:

$$\psi = \frac{C(a-N)^{1-(2\alpha-1)b(a-b)^{-1}}(N-b)^{1+(2\alpha-1)a(a-b)^{-1}}}{N^{2\alpha}}. \tag{3.6}$$

Here a and b are the roots of the square equation $N^2 + AB_0/2N + 1/2 = 0$ $(a > b)$ and C is an arbitrary constant. We consider that $4D = A^2B_0 - 8 > 0$. The boundary conditions for the function ψ are satisfied at the points $N = a$ and $N = b$ provided that

$$1 - (2\alpha - 1)b/(a-b) > 0, \qquad 1 + (2\alpha - 1)a/(a-b) > 0. \tag{3.7}$$

Let $y = 0$ correspond to $N = a$, and $y = 1$ to $N = b$. Since $L = \psi'_y = \psi'_N N'_y$, from (3.6) we obtain

$$y'_N = -C(a-N)^{-(2\alpha-1)b(a-b)^{-1}}(N-b)^{(2\alpha-1)a(a-b)^{-1}}/N^{2\alpha+1}. \tag{3.8}$$

Since $N^2 = \psi' + \alpha = B_0^{-1}(U-A)^{-2}$, the quantity $(U-A)^{-1}$ varies from $\sqrt{B_0}a$ for $y = 0$ to $\sqrt{B_0}b$ for $y = 1$. Integrating Eq. (3.8), we obtain a relation which connects the horizontal velocity U and the coordinate Y:

$$\frac{Y}{A_0} = \frac{\displaystyle\int_0^{(a-b)^{-1}(ab\sqrt{B_0}(U-A)-b)} z^{-(2\alpha-1)b(a-b)^{-1}}(1-z)^{(2\alpha-1)a(a-b)^{-1}}\,dz}{\displaystyle\int_0^{1} z^{-(2\alpha-1)b(a-b)^{-1}}(1-z)^{(2\alpha-1)a(a-b)^{-1}}\,dz}. \tag{3.9}$$

According to (3.2), formula (3.9) determines the velocity profile in a simple wave $u = u(p(x,t), Y)$. The velocity varies from $u = u_1 = (\sqrt{1-\beta})^{-1}p^{(1-\beta)/2}(A + (\sqrt{B_0}a)^{-1})$ at the lower wall $Y = 0$ to $u = u_2 = (\sqrt{1-\beta})^{-1}p^{(1-\beta)/2}(A + (\sqrt{B_0}b)^{-1})$ at the upper wall $Y = A_0$. To satisfy the inequalities (3.7), it suffices to

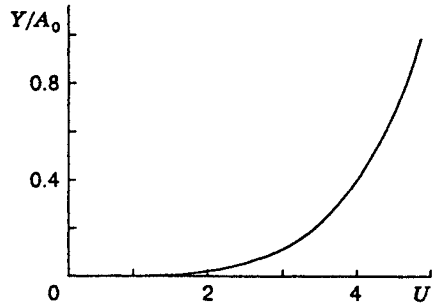

Fig. 1

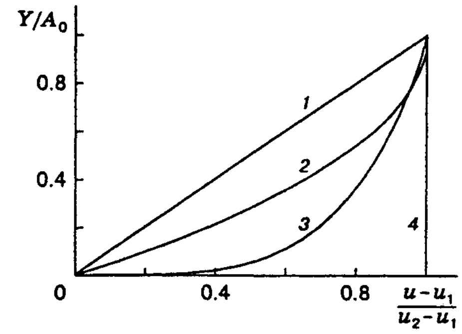

Fig. 2

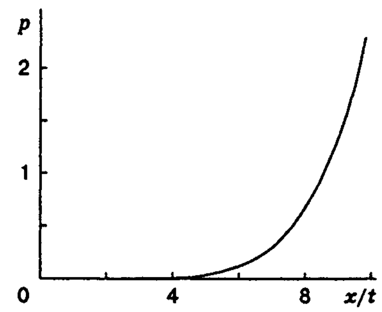

Fig. 3

require $A^2 > (2\alpha + 1)^2(\alpha B_0)^{-1}$. If $A > 0$, we have $k > u_{\max} = u_2$ and the simple wave faces to the right, whereas if $A < 0$, we have $k < u_{\min} = u_1$ and the wave faces to the left. For $\beta = 1/3$, relation (3.9) is integrated in quadratures: $U = A + 2(a - b)(A_0\sqrt{B_0})^{-1}Y + (a\sqrt{B_0})^{-1}$.

The case $D = 0$ ($A^2B_0 = 8$) corresponds to a shear-free flow: $u = (\sqrt{1-\beta})^{-1}(A + (\sqrt{B_0}\alpha)^{-1})$ and $v = 0$. The contact surfaces in this case are rectilinear: $Y(\lambda, p) = \text{const} = Y(\lambda, p_0)$.

Figure 1 shows the horizontal-velocity profile for $\beta = 2/3$, $A = 5$, and $B_0 = 3$. Formula (3.9) defines Y/A_0 as a function of $(u - u_1)/(u_2 - u_1)$. Figure 2 illustrates the diagrams of Y/A_0 versus $(u - u_1)/(u_2 - u_1)$ for $|A| = 5$, $B_0 = 3$, and $\beta = 1/3$, $1/5$, and $2/3$ (curves 1–3, respectively). Vertical straight line 4 corresponds to a shearless flow ($A^2B_0 = 8$).

In the case of a centered simple wave where $k = x/t$, from (3.1) and (3.7) we obtain the pressure profile in a simple wave $p(x,t) = ((1 - \beta)A^{-2})^{(1-\beta)^{-1}}(x/t)^{2(1-\beta)^{-1}}$. The diagram of the pressure distribution for $\beta = 2/3$ and $A = 5$ is shown in Fig. 3.

As a result, it has been shown that a simple wave can be adjacent, in its characteristic, to any shear flow and it is either a compression or rarefaction wave, depending on the monotonicity properties of the function $k(p)$. In a medium with constant entropy, a simple rarefaction wave exists for all $t > 0$, and the compression wave decays. A class of exact solutions of a system of long-wave equations that describe simple waves propagating in a barotropic gas with velocity k has been found.

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